Bubble Oscillations

Consider a spherical bubble of radius R, of a certain fluid of density ρ, trapped inside of some other fluid. The bubble is stabilized by the presence of surface tension. Namely, suppose that the bubble has a nearly, but not perfectly, spherical surface, which we describe by a function $\zeta(\theta, \phi)$, denoting the difference $\zeta = r - R$ between the actual radius r and the original radius R. One can then write the energy cost of this deformation as

$$E = \alpha \int \mathrm{d}\theta \mathrm{d}\phi \sin\theta (R+\zeta)^2 \sqrt{1 + \left(\frac{1}{R+\zeta}\frac{\partial \zeta}{\partial \theta}\right)^2 + \left(\frac{1}{(R+\zeta)\sin\theta}\frac{\partial \zeta}{\partial \phi}\right)^2}.$$

(a) Argue that the pressure (difference from equilibrium) at the surface of the bubble

$$P = \frac{2\alpha\zeta}{R^2} + \frac{\alpha}{R^2}\nabla^2\zeta$$

where ∇^2 is the appropriate angular Laplacian, in spherical coordinates.

(b) Argue that the fluid flow inside the bubble is incompressible, and can be described by a velocity potential ψ. Determine an equation for the boundary condition at $r \approx R$ for ψ by relating ψ to ζ, and using the result from earlier.

(c) Show that the eigenmodes are described by spherical harmonics (l, m), and that the oscillation frequencies of the fluid, and thus of the bubble's surface as well, are given by

$$\omega_{lm} = \sqrt{l(l-1)(l+2)\frac{\alpha}{\rho R^3}}.$$

(d) Give a physical reason for why the $l = 0$ and $l = 1$ modes do not oscillate. Sketch the bubble surface's motion for the $(2, 0)$ mode.