Train Track Waves

One of the "dangers" of high speed rail can be the propagation of waves along the train tracks. For one thing, this may cause damage to the tracks, but also it will radiate energy away, thus requiring the train to spend more energy to move itself forward. In this problem, we will consider the following crude model for this effect.

Let $y(x,t)$ be the transverse displacement at position x along the track, at time t. The wave equation for y, including the effect of a train moving along the track at velocity v, can crudely be modeled by

$$\alpha\partial_x^4 y + ky + \lambda\partial_t^2 y = P\delta(x - vt),$$

with α, k, λ and P all positive constants.

(a) Make the ansatz that the solution is of the form $y(x - vt)$, and then reduce the wave equation to an ODE.

(b) Discuss what happens as the speed v increases. Show that there is a critical speed v_c such that, when $v \geq v_c$, the form of the solutions will radically change.

(c) Find the explicit solution to the wave equation when $v < v_c$.

(d) Find the explicit solution to the wave equation when $v > v_c$, and show that in this case energy will be radiated away from the train.