Vortices in the Wave Equation

When (random) wave trains are scattered off of (random) surfaces, it is common to observe the formation of topological defects in the scattered waves: i.e., dislocations or vortices. Let us consider a complex scalar field obeying the wave equation

$$c^2\nabla^2\psi = \frac{\partial^2\psi}{\partial t^2}.$$

For example, this can be used in some cases to study electromagnetic waves. Let us work in 2 spatial dimensions.

Let us write

$$\psi = \rho \mathrm{e}^{\mathrm{i}\chi}.$$

$\rho \ge 0$, and χ, are real-valued. Assume that ψ is smooth. A topological defect in ψ is a point where the field χ is ill-defined.

(a) Show that topological defects occur when $\rho = 0$. Conclude that topological defects can be classified by a "winding number" n, which is an integer: if the defect occurs at $x = y = 0$, and θ is a polar coordinate angle around this point, then[1]

$$\chi = n\theta + F(\theta), \text{ where } F \text{ is periodic.}$$

(b) Verify that this is a solution:

$$\psi = A\left[(1 - B(k_x x + k_y y - \omega t))\,\mathrm{e}^{\mathrm{i}(k_x x + k_y y - \omega t)} - (1 - B(k_x x - k_y y - \omega t))\,\mathrm{e}^{\mathrm{i}(k_x x - k_y y - \omega t)}\right],$$

if $\omega = ck$, where $k = |\mathbf{k}|$. Take $A, B > 0$.

(c) For this choice of ψ, determine the fields ρ and χ. For simplicity, assume that $Bk_y y \ll 1$. Determine the location of vortices (they're here!) as a function of (x, y, t), and (carefully) determine the winding number of each vortex.

(d) How fast do the vortices move?

(e) Let $\mathbf{x}(t)$ describe the position of a vortex as a function of time; in a moving coordinate system moving along with the vortex, let $\delta\mathbf{x} \equiv \mathbf{x} - \mathbf{x}(t)$. Determine ρ and χ to leading order in δx and δy. Numerically plot lines of constant χ around this vortex – what do you see?[2]

[1] Try plugging in this ansatz, making no assumptions about n. Remember ψ must be well-behaved!

[2] For anyone familiar with the theory of dislocations in crystals, this should look familiar!