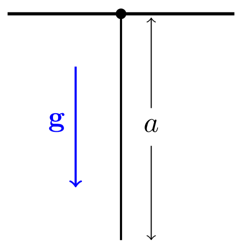
Wave on a Hanging String

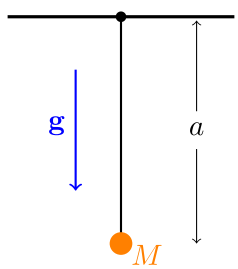
A string has a mass m, uniformly distributed along its length a. It is fixed at one end to the ceiling, and allowed to hang freely at the other end in a gravitational field of strength g. Let the displacement of the string be $u(z,t) \ll a$; the approximations that were made in deriving the wave equation on a string previously are still valid.



(a) Write down the wave equation for this string.

(b) Find the normal modes $U_n(z)$ and the associated eigenfrequencies ω_n. To do this, let $x = \sqrt{z}$: the ODEs for U_n should look much more familiar.

Suppose we attach a point mass M to the bottom of the string.



(c) While the normal modes $U_n(z)$ and the eigenfrequencies ω_n do not have the simple expressions that they did in (b), at least write down a set of (non-differential) equations which, upon solution, give the normal modes and eigenfrequencies.

(d) Show that in the limit $M \gg m$, the functions reduce to something familiar. Be sure to justify physically why this happens.

You have just solved the dynamics of a pendulum with a massive string (in the small amplitude approximation)!