Kelvin Waves

A line vortex has radius a, such that the velocity field for $r \geq a$ is

$$\mathbf{v}_0 = \frac{\gamma}{2\pi s}\hat{\boldsymbol{\phi}}.$$

Now, suppose we allow the boundary of the core to vibrate slightly, so that the radius is given by $a + a_1$, where $|a_1| \ll a$. Let the perturbations to pressure and velocity potential be P_1 and φ_1.

(a) Show that, to first order, the boundary conditions at $r = a$ become

$$\frac{\partial a_1}{\partial t} + \frac{\gamma}{2\pi a^2}\frac{\partial a_1}{\partial \phi} = -\frac{\partial \varphi_1}{\partial r},$$
$$\frac{\partial \varphi_1}{\partial t} + \frac{\gamma}{2\pi a^2}\frac{\partial \varphi_1}{\partial \phi} = -\frac{\gamma^2}{4\pi^2 a^3}a_1.$$

(b) Show that if φ_1 and a_1 go like $\mathrm{e}^{\mathrm{i}(m\phi + kz - \omega t)}$,

$$\omega(k) = \frac{\kappa}{2\pi a^2}\left(m \pm \sqrt{g_m(ka)}\right)$$

where the function $g_m(x)$ is defined as

$$g_m(x) = -\frac{x\mathrm{K}_{|m|}(x)}{\mathrm{dK}_{|m|}(x)/\mathrm{d}x}.$$

(c) Let $m = -1$. Draw/describe the motion of the core, and show that

$$\omega(k) \approx \frac{\gamma}{4\pi}\left[0.116 - \log(ka)\right]k^2.$$