LSD

In a series of biology papers from the 1970s, certain strange patterns are "reported to occur" among LSD users in their hallucinations. A very simple mechanism is likely to lead to these strange patterns. The basic idea is as follows. Suppose that there is some field $\phi(x, y, t)$ which obeys some complicated nonlinear partial differential equation:

$$\partial_t \phi = D\nabla^2\phi - E\phi\nabla^4\phi + \cdots$$

We take $\phi(x, y)$ to represent rates of firing neurons, for example, in the cortical plane. Suppose this equation has an instability for some very small rage of wave vectors $k \approx k_c$. We will assume that solutions to this PDE which are time-independent are superpositions of plane waves $\exp[\mathrm{i}(k_x x + k_y y)]$ where $\sqrt{k_x^2 + k_y^2} = k_c$. This is a reasonable approximation for certain classes of nonlinear PDEs.

Now, this image of firing neurons is "observed by the eye" as follows (in reality, it is the case that the neurons which fire are the ones which correspond to the observation of images from the eye...). If (r, θ) are polar coordinates in the retinal plane (this plane corresponds to the effective images we observe), then we find

$$\begin{aligned} x &= \log r, \\ y &= \theta. \end{aligned}$$

The "intensity of light" in our "image" corresponding to the hallucination is

$$\phi_{\text{observed}}(r, \theta) = \phi(x(r, \theta), y(r, \theta)).$$

(a) In order for the image on the retinal plane to be well-defined, what are the restrictions on k_y?

(b) Explore what happens for various possibilities of k_x and k_y numerically, if you take ϕ to consist of only a *single* plane wave with fixed k_x and k_y. Print out contour plots showing all possible behaviors, and give explanations for why each occurs.

(c) A rectangular combination of plane waves is a combination of plane waves which is invariant under $\pi/2$ rotations of (k_x, k_y). Explain why there is countably infinite class of unique LSD rectangular waves which are well-defined, and show some sample contour plots, obtained numerically. You should find that all of them are qualitatively similar.

(d) A hexagonal combination of plane waves is a combination of plane waves which is invariant under $2\pi/3$ rotations of (k_x, k_y). Explain why there is a unique well-defined hexagonal pattern for LSD, and print out a contour plot of what you find. The pattern you find is exotic, and you do not need to explain your result.