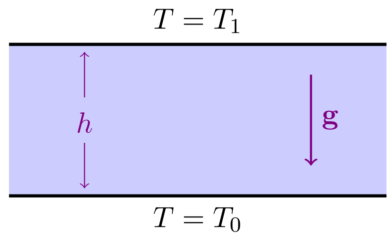
Rayleigh-Bénard Convection

Two plates are held fixed a distance h apart, but are allowed to slide within the plane freely.[1] At $z = 0$, the temperature of the plate is $T = T_0$; at $z = h$, $T = T_1$. In between, we have a roughly incompressible fluid of density ρ_0, kinematic viscocity ν, thermal diffusivity κ and thermal expansion coefficent β, in a gravitational field $\mathbf{g} = -g\hat{\mathbf{z}}$.



As in most fluids, we have $\beta(T_1 - T_0) \ll 1$.

(a) Accounting for non-constant ρ due to thermal expansion, show that the first-order correction to the Navier-Stokes equation is

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{1 + \beta(T - T_0)}{\rho_0} \nabla P - g(1 - \beta(T - T_0))\hat{\mathbf{z}} + \nu \nabla^2 \mathbf{v}.$$

The other two equations are mass conservation, which is approximately

$$\nabla \cdot \mathbf{v} = 0,$$

and the diffusion equation for temperature:

$$\frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T = \kappa \nabla^2 T.$$

(b) Find a static solution with $T = T_0(z)$, $P = P_0(z)$, and $\mathbf{v} = \mathbf{v}_0 = \mathbf{0}$.

In general, the solution will be of the form $T = T_0 + T_1$, $P = P_0 + P_1$, and $\mathbf{v} = \mathbf{v}_1$. As the solution will be lengthy, we will first make the parameters of the problem dimensionless by the following re-labeling:

$$\tilde{\mathbf{r}} = \frac{\mathbf{r}}{h},$$
$$\tilde{t} = \frac{\nu}{h^2} t,$$
$$\tilde{\mathbf{v}} = \frac{h}{\nu} \mathbf{v},$$
$$\tilde{P} = \frac{h^2}{\rho_0 \nu^2} P_1,$$
$$\tilde{T} = \frac{\kappa}{\nu(T_1 - T_0)} T_1.$$

[1] This is an unlikely boundary condition for a real fluids problem, but you'll find the problem much harder to solve with a more realistic fixed plate condition!

(c) Show that (dropping tildes and subscripts), our equations become

$$\nabla \cdot \mathbf{v} = 0,$$
$$\frac{\partial \mathbf{v}}{\partial t} = -\nabla P + \nabla^2 \mathbf{v} + RT\hat{\mathbf{z}},$$
$$Q\frac{\partial T}{\partial t} = \nabla^2 T + v_z$$

where Q is the Prandtl number and R is the Rayleigh number:

$$Q \equiv \frac{\nu}{\kappa},$$
$$R \equiv \frac{gh^3\beta(T_1 - T_0)}{\kappa\nu}.$$

(d) What are the boundary conditions in this problem?

We are interested in solutions of the form $T(\mathbf{r}, t) = \mathrm{e}^{\gamma t + \mathrm{i}kx}T(z)$ (similar equations hold for P and $\mathbf{v}$) for the perturbations around the static solution.

(e) Prove that γ is real. To do this, look at $\partial_t|\mathbf{v}|^2$ and $\partial_t|T|^2$.

(f) Reduce the dimensionless equations to a single 6^{th} order ODE for $T(z)$.

(g) Show that, for $n \in \mathbb{N}$, if

$$T(z) = \mathrm{e}^{\gamma t + \mathrm{i}kx}\sin(n\pi z),$$

then $T(z)$ satisfies the boundary conditions (for all fields!) if

$$\gamma = -\frac{l^2}{2}\left(1 + \frac{1}{Q}\right) \pm \sqrt{\left(\frac{l^2}{2}\left(1 - \frac{1}{Q}\right)\right)^2 + \frac{Rk^2}{Ql^2}}.$$

where

$$l^2 = k^2 + n^2\pi^2.$$

Obviously, when $\gamma > 0$, there is instability in the system. This instability is called **convection** and, along with conduction and (electromagnetic) radiation is one of the three classical methods of heat transfer studied in elementary science courses. This mechanism for convection was discovered by Rayleigh and Bénard.

(h) Show that the critical point in (k, R) at which we observe convection is given by

$$k_{\mathrm{c}} = \frac{\pi}{\sqrt{2}} \approx 2.22,$$
$$R_{\mathrm{c}} = \frac{27\pi^4}{4} \approx 658.$$

(i) Draw the velocity field $\mathbf{v}$ when there is convection. Comment on the trajectories of test particles.

Velocity patterns similar to the one of part (i) appear in real fluids in convection, although they are far more complex and two-dimensional. In order to find these patterns, the full nonlinear PDEs must be approximated.