Sound of a Turbulent Flow

Suppose you are an observer sitting very far away from a region of space with turbulent fluid flow, which is (statistically) constant in time. In particular, if you are a distance r away from the region, we require that $r^3 \gg V$, where V is the volume of space with turbulent flow. Generically, the flow of a turbulent fluid will generate sound waves. Let us denote with $\mathbf{v}(\mathbf{x}, t)$ the velocity of the fluid flow in the turbulent region.

(a) Following the derivation of the equation of motion for sound waves in the absence of background fluid flow, show that in the presence of the background turbulent flow, the equations of motion can be approximated by

$$\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} - \nabla^2 p = \frac{\partial^2}{\partial x_i \partial x_j}(\rho v_i v_j).$$

Here we are letting $p = P - P_0$ be the deviation of pressure from the background, and c is the speed of sound.

(b) Show that

$$p(r,t) \approx \frac{1}{4\pi r c^2}\frac{\partial^2}{\partial t^2}\int \mathrm{d}V \quad \rho \mathbf{v}^2\Big|_{\text{ret. time}}.$$

These equations are true in general. But now, we want to average over turbulent flow. In turbulent flow, we are most interested in the average acoustic intensity, given by

$$I = \frac{\langle p^2 \rangle}{\rho c}.$$

In turbulent flow, the system is characterized by many "eddys" of size L, with fluid flow at velocity v in each region. The characteristic time scale of such an eddy is then $T = L/v$. The flows in different eddys are, for the most part, uncorrelated with each other.

(c) Using dimensional analysis, estimate I up to an overall dimensionless pre factor.

(d) In turbulent flow, it is often more convenient to express things in terms of the Mach number $M = v/c$, and in terms of the turbulent dissipation rate ϵ, which has units of power per mass. Show that

$$\frac{r^2 I}{V} \sim \rho \epsilon M^5.$$

Thus, turbulent flow radiates energy away with the exponent M^5.